

Synchronization of N-Nonlinear Chaotic or Complex Systems by Feedback Controller

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Abstract- In this paper, our core purpose is to study the synchronization of two different systems by the feedback controller technique. These two systems are dynamically nonlinear systems having different time delays in the presence of disturbance. The master state of the system has the energy-bounded input noise. This proposed technique is exceptionally advanced. In this scheme, it can synchronize the nonlinear time-delayed master and slave system very precisely with free of error. The synchronizing time and the minimization of the error rate depend upon the gain factor of the controller. The applicability and effectiveness of the given controller topology are verified by the simulation example of phase-locked loops in our research paper. The simulation results witness that the error converges to origin with time. The systems get synchronized after the error becomes zero between the states of the master and slave system.

Index Terms - Synchronization, Output feedback, Complex systems

I. INTRODUCTION

Synchronization can be defined in terms of more than two periodic or dynamical systems that are emerged each other or coupled together, as discussed in [1]. In different fields such as Mathematics, Applied Physics, Control Science, and Biology, this concept is widely used. The concept of synchronization for the chaotic and complex systems has received a significant focus in this modern era's research. Several methods of synchronization techniques were introduced based on the control approach. These various techniques such as the observer-based control approach as discussed in [2,3], adaptive control technique in [4], sliding mode controller approach elaborated in the [5], time-delay feedback control scheme is in [6], impulsive controller scheme is discussed in [7] and the great inspiring work of Carroll and Pecora was discussed in [8]. These are the most important and widely used techniques in the control field such as the synchronization of any systems. Apart from the core motivation, this is widely used in the applications of securing communications. The numerous forms of chaotic synchronization include synchronization of the Lur'e master and slave system. The work for the synchronization of chaotic Lur'e system was controlled in different ways. The absolute stability theory in [9] and different circumstances have been established in [10-11]. Nevertheless, there exists the study of nonlinear systems with numerous symmetries and generalization of mathematical equations of the pendulum model. This pendulum system is a kind of multi equilibria approach which is practically

used in different fields such as mechanical and electrical engineering applications (phase-locked loop and synchronous machine). The objective of this research paper is to synchronize the unbalanced master pendulum system and slave system by robust feedback technique and LMI based method for the synchronization of the chaotic dynamical pendulum system and output feedback controller technique. The closed-loop error minimizes after very little time and the system becomes stable, so the disturbance input effect reduces. To validate our research results we have taken the example of the phase-locked loop system.

II. SYSTEM DESCRIPTION

The nonlinear feedback controller systems, having infinite equilibria and periodic nonlinearity cover an extensive type of system in mechanics, engineering, power systems, and different fields. The synchronization is commonly used in the engineering field to achieving the synchronization between the two master and slave systems. We have taken the state space representation of these systems as given below in equations (1) and (2) respectively and by using the feedback controller scheme we can get the desired output. We will consider the following nonlinear feedback controller systems in our research paper:

$$\dot{X}_m = AX_m + A_d X_m(t-\tau) + B\beta(\Upsilon_1) + \omega, \quad (1)$$

$$h_1 = \dot{\Upsilon}_1 = C^T X_m + C_d^T X_m(t-\tau) + D\beta(\Upsilon_1),$$

$$\dot{X}_s = AX_s + A_d X_s(t-\tau) + B\beta(\Upsilon_2) + u, \quad (2)$$

$$h_2 = \dot{\Upsilon}_2 = C^T X_s + C_d^T X_s(t-\tau) + D\beta(\Upsilon_2),$$

In both system states (1) and (2) the states vectors such as $X_m \in R^n$ and $X_s \in R^n$. Whereas different constant real matrices are the following as $A_d, A \in R^{n \times n}$, $B \in R^{n \times l}$, $C, C_d \in R^{m \times n}$ and $D \in R^{m \times m}$. The output vectors are written in the form as $h \in R^m$, the ω is the disturbance and the phase vector is $\Upsilon \in R^m$. The u variable is the unidirectional-coupled term, the specific objective of this variable is to control the output. The continuously differentiable vector $\beta: R^m \rightarrow R^m$ and valued functions are $\beta(\Upsilon_n)$. This component is considered as periodic and finite numeral of zero's in time intervals. Master system can be characterized by the transfer function of linear part from input $\beta(\Upsilon)$ to output $\dot{\Upsilon}$ and the subsequent supposition was made on the equation (1) with phase variable Υ . The Transfer function $T(s)$ was hypothetically taken as a non-singular matrix.

$$T(s) = C^T (SI - A)^{-1} B - D$$

The pairs like (A, B) are the controllable matrix and the observable parameters are (A, C^T) .

$$A(x_{eq}) = -B\beta(\Upsilon_{eq}), \quad \left(C^T (A)^{-1} B - D \right) \beta(\Upsilon_{eq}) = 0$$

The parameter $\beta(\Upsilon_{eq}) = 0$ and we know the equation (1) of the system has periodicity and infinitely isolated stabilities. In this paper, our main objective is to synchronize the given systems (1) and (2) by using the fast controller scheme. Synchronization can be efficiently achieved in the presence of noise between the master and the slave system. The H_∞ is the technique in which performance is employed by the feedback controller to the slave system by the help of the controller signal $B \in R^{n \times l}$. The error signal exists between the master and the slave system, these errors are defined as $\lambda = h_1 - h_2$. Synchronization error dynamics among the chief and the slave system is elaborated in equation (3) as given below:

$$\begin{aligned} \dot{\varepsilon} &= \dot{X}_m - \dot{X}_s \\ \dot{\varepsilon} &= A\varepsilon + A_d \varepsilon_d + B(\beta(\Upsilon_1) - \beta(\Upsilon_2)) - u + \omega, \end{aligned} \quad (3)$$

$$\dot{\lambda} = C^T \varepsilon + C_d^T \varepsilon_d + D(\beta(\Upsilon_1) - \beta(\Upsilon_2)),$$

Here the term $(\beta(\Upsilon_1) - \beta(\Upsilon_2))$ is the periodic function.

III. SYNCHRONIZATION VIA H_∞ CONTROLLER

Error dynamics during synchronization of master and slave systems, a controller can be used for synchronization if diminution disturbance exists and this can satisfy the two conditions where the first condition is zero initial condition and γ is greater than zero. The second condition is zero disturbances where input is exponentially stable; this controller is called H_∞ controller and γ is called H_∞ -norm bound of the controller. This controller can synchronize the master and slave system and the controller is given as below:

$$G = \int_0^\infty \left[\lambda^T \lambda - \gamma^2 \omega^T \omega \right] dt, \quad (4)$$

$$G \leq 0,$$

IV. SYNCHRONIZATION VIA FEEDBACK CONTROLLER

The master and slave are models of the pendulum-like system. It can be synchronized by matrix inequality-based approach under the robustly synchronizing and feedback controller u which is given below as:

$$u = K \begin{pmatrix} \dot{\Upsilon}_1 - \dot{\Upsilon}_2 \end{pmatrix}, \quad (5)$$

The u is the control input and we will put this u value in the equation (3). The gain constant is $K \in R^{n \times m}$ and in this paper, we have used the feedback scheme, this can fast and slow the system response, and synchronization error dynamics can be rewritten as

$$\begin{aligned} \dot{\varepsilon} &= (A - KC^T) \varepsilon + (A_d - KC_d^T) \varepsilon_d \\ &\quad + (B - KD)(\beta(\Upsilon_1) - \beta(\Upsilon_2)) + \omega, \\ &= A_o \varepsilon + A_{od} \varepsilon_d + B_o (\beta(\Upsilon_1) - \beta(\Upsilon_2)) + \omega, \\ \dot{\lambda} &= C^T \varepsilon + C_d^T \varepsilon_d + D(\beta(\Upsilon_1) - \beta(\Upsilon_2)), \end{aligned} \quad (6)$$

The system with zero initial condition and diagonal matrix can be defined in term of $v = \text{diag}(v_1, \dots, v_4)$.

$$v_i = \frac{\int_0^{T_i} (\beta(\Upsilon_i) - \beta(\Upsilon_{2i})) d\beta}{\int_0^{T_i} |(\beta(\Upsilon_i) - \beta(\Upsilon_{2i}))| d\beta}, \quad i = 1, 2, 3 \dots n, \quad (7)$$

$$F_i \lambda_i = (\beta(Y_i) - \beta(Y_{2i})) - v_i \left| (\beta(Y_i) - \beta(Y_{2i})) \right|,$$

and satisfies the $\int_0^{T_i} F \lambda_i d\lambda = 0$ and after satisfying this function, master and slave system becomes synchronized.

Theorem: The master and slave system synchronize with the help of a robust H_∞ controller. Equation (8) can be termed as H_∞ synchronization controller that has disturbance attenuation γ . Conditions for synchronization are symmetric matrix $W > 0, M, N$ and a diagonal matrix $\kappa, l, \sigma > 0$. As explained in following matrices

$$\begin{bmatrix} \Theta_1 & \Theta_2 & W + MA^T N^T + MA_d^T N^T - CK^T N^T - C_d K^T N^T & -M & C\ell \\ \otimes & \Theta_3 & -B^T N^T - D^T K^T N^T & 0.0 & -M \\ \otimes & \otimes & N + N^T & N & 0.0 \\ \otimes & \otimes & \otimes & i^2 I & 0.0 \\ \otimes & \otimes & \otimes & \otimes & -\ell \end{bmatrix} < 0 \quad (8)$$

$$\begin{bmatrix} 2l & \kappa v \\ \oplus & 2\sigma \end{bmatrix} > 0 \quad (9)$$

$\Theta_1, \Theta_2, \Theta_3$ these are further explained as,

$$\Theta_1 = -(A^T + A_d^T + CK^T + C_d K^T)M^T - M(A + A_d + KC^T + K_d C^T) + CC^T$$

$$\Theta_2 = \frac{1}{2}(C + C_d) + CD + C_d D - M - MKD$$

$$\Theta_3 = \sigma + D^T D + 1/2 D^T \kappa + 1/2 \kappa D$$

Proof:

The concept of a Lyapunov function is used in this paper.

$$V_1(\varepsilon, \lambda) = \varepsilon^T W \varepsilon, \quad V_2(\varepsilon, \lambda) = \sum_{i=1}^n \kappa_i \int_0^{\lambda_i} F_i(X_s) dx$$

$$V(\varepsilon, \lambda) = V_1(\varepsilon, \lambda) + V_2(\varepsilon, \lambda)$$

$$V(\varepsilon, \lambda) = \varepsilon^T W \varepsilon + \sum_{k=1}^n \kappa_k \int_0^{\varepsilon_k} F_k(X_s) dx$$

For matrices such as M, N with suitable dimensions,

$$\Delta = \begin{bmatrix} \varepsilon^T & \dot{\varepsilon}^T \end{bmatrix} \begin{bmatrix} M & 0.0 \\ 0.0 & N \end{bmatrix} \begin{bmatrix} \varepsilon - A_o \varepsilon - A_{od} \varepsilon_d - B_o (\beta(Y_1) - \beta(Y_2)) - \omega \\ \dot{\varepsilon} \end{bmatrix}$$

$$\Delta = 0$$

By incorporating term 2Δ , and taking time derivative of these terms $V(\varepsilon, \lambda)$, $V_1(\varepsilon, \lambda)$ and $V_2(\varepsilon, \lambda)$ are as given below.

$$\dot{V}_1(\varepsilon, \lambda) = 2\varepsilon^T W (A_o \varepsilon + A_{od} \varepsilon_d + B_o (\beta(Y_1) - \beta(Y_2)) + \omega) + 2\Delta$$

$$\dot{V}_2(\varepsilon, \lambda) = \sum_{i=1}^n \kappa_i F_i(\lambda_i) \dot{\lambda}_i$$

$$\dot{V}(\varepsilon, \lambda) = \dot{V}_1(\varepsilon, \lambda) + \dot{V}_2(\varepsilon, \lambda)$$

$$= \sum_{i=1}^n \left[k_i \dot{\lambda}_i^2 - \sigma_i \beta^2(Y_i - Y_{2i}) \right] + \sum_{i=1}^n \left[\kappa_i (\beta(Y_i) - \beta(Y_{2i})) \dot{\lambda}_i - \kappa_i v_i \left| (\beta(Y_i) - \beta(Y_{2i})) \right| \dot{\lambda} \right] - \ell_i \dot{\lambda}_i^2 - \sigma_i (\beta^2(Y_i) - \beta^2(Y_{2i}))$$

It can be considered from (9) that

$$\begin{aligned} & \ell_i v_i \left| (\beta(Y_i) - \beta(Y_{2i})) \right| \dot{\lambda}_i + \ell_i \dot{\lambda}_i^2 + \sigma_i (\beta^2(Y_i) - \beta^2(Y_{2i})) \\ & \geq \ell_{0i} \dot{\lambda}_i^2 + \delta_{0i} (\beta^2(Y_i) - \beta^2(Y_{2i})) \end{aligned}$$

Let

$$\dot{\varphi} = \dot{\varepsilon} = A_o \varepsilon + A_{od} \varepsilon_d + B_s (\beta(Y_i) - \beta(Y_{2i})) + \omega,$$

$$\dot{\phi} = \dot{\lambda} = C^T \varepsilon + C_d^T \varepsilon_d + D (\beta(Y_i) - \beta(Y_{2i}))$$

$$\dot{V}(\varepsilon, \lambda) + \sum_{k=1}^m \left[\ell_{0k} \dot{\xi}_k^2 + \delta_{0k} \eta_k^2(\ell_k, Y_{2k}) \right] + \lambda^T \lambda - i^2 \omega^T \omega$$

After further calculations we arrive on the result that,

$$= \begin{bmatrix} \varepsilon^T & (\beta^T(Y_1) - \beta^T(Y_2)) & \varphi^T & \omega^T \end{bmatrix} \rho \begin{bmatrix} \varepsilon^T & (\beta^T(Y_1) - \beta^T(Y_2)) & \varphi^T & \omega^T \end{bmatrix}^T$$

$$\rho = \begin{bmatrix} \rho_1 & \rho_2 & W + M - A^T N^T - A_{od}^T N^T & -M \\ \otimes & \rho_3 & -B_s^T N^T & 0.0 \\ \otimes & \otimes & N + N^T & -N \\ \otimes & \otimes & \otimes & -i^2 I \end{bmatrix} \quad (10)$$

$$\begin{aligned} \rho_1 &= -(A_o^T M^T + A_d^T M^T + MA_o + MA_{od}) \\ &\quad + C\ell C^T + C_d C^T + CC^T, \end{aligned}$$

$$\rho_2 = 1/2(C\kappa + C_d) + C\ell D + C_d D + CD - MB_o,$$

$$\rho_3 = \sigma + D^T \ell D + D^T D + 1/2 D^T \kappa + 1/2 \kappa D,$$

The system (3) with $\omega = 0$ is represented as follows. Upper left block of matrix Π as shown above is also negative semi definite matrix that represents.

$$\begin{bmatrix} A_o^T T^T + TA_o + A_{od}^T T^T + TA_{od} + C\ell C^T & 1/2(C\kappa + C_d) + C\ell D + TB_s \\ \otimes & \sigma + D^T \ell D + 1/2D^T \kappa + 1/2\kappa D \end{bmatrix} \quad (11)$$

$$+ \begin{bmatrix} (C + C_d) \\ D^T \end{bmatrix} \begin{bmatrix} (C + C_d)^T & D^T \end{bmatrix} \leq 0$$

By combining the condition of equation (11) and integration of equation (10) results are obtained as,

$$V(t) - V(0) \leq - \sum_{k=1}^n \int_0^t \left[\ell_{0i} \dot{\lambda}_i^2 + \delta_{0k} \beta(Y_i - Y_{2i}) \right] dt$$

Because of the stable conditions, we noticed that the solution ε of (3) is bounded, and the boundless form of equation $\int_0^\lambda F_k(X) dx$ is determined by the fact that $F_i(\lambda_i)$ has a mean value of zero that demonstrates that $V(t)$ is bounded. By combining the above expressions, the following results are obtained for $i = 1, 2, \dots, n$:

$$\begin{aligned} \int_0^t \sigma_{0i} \beta^2(Y_i(t) - Y_{2i}(t)) dt &< +\infty \\ \int_0^t \lambda_k^2(t) dt &< +\infty, \int_0^t \varepsilon_i^2(t) dt < +\infty \end{aligned} \quad (12)$$

Since the solutions for taking the master and slave system is bounded, nonlinearities factor is $\beta^2 Y_i(t) - \beta^2 Y_{2i}(t)$,

$i = 1, 2, \dots, n$ are uniformly continuous. Moreover, the only β factor is periodic function and has finite number of zero's we have

$$\begin{aligned} \lim_{t \rightarrow \infty} (\beta_i(Y_i) - \beta_i(Y_{2i})) &= 0, \\ \lim_{t \rightarrow \infty} \lambda_i(t) &= \hat{\lambda}_i(t), \quad i = 1, 2, \dots, n \end{aligned} \quad (13)$$

Above mechanism explains that equation (6) is satisfied under zero initial conditions for all nonzero terms. Since $\rho < 0$ for all ε, l and ω satisfying equation (4), it can be reduced by equation (10) in such a way that $\square V(\varepsilon, \lambda) + \rho^T \rho - \gamma^2 \omega^T \omega < 0$ which shows the robust synchronization of master slave system.

V. SIMULATION AND RESULTS

We taking the example of a phase-locked loop system to elaborate our research result. This example is commonly used in electrical engineering and other engineering fields such as used in demodulators, retiming devices, and frequency synthesizers [12,13]. We are taking the two phase-locked loop interconnected systems which behave like the master system and slave system, these two systems are nonlinear, whereas the dynamics of systems are illustrated in the equation (1). These systems also have an external disturbance. The transfer function obtained from the ratio of output to input of the system is shown as $T(s) = \frac{1}{s} X(s)$. Here,

$X(s)$ is represented by the state-space equations. The state-space model is described as given below:

$$\begin{aligned} \square x &= \begin{bmatrix} 0.2 & -0.45 & 0.3 \\ 4 & -0.5 & 0.4 \end{bmatrix} x + \begin{bmatrix} -0.1 \\ 0.1 \\ 0.2 \end{bmatrix} \cos(90 - Y), \\ \square Y &= \begin{bmatrix} .2 \\ 2 \\ .3 \end{bmatrix} x + 0.4 \cos(90 - Y), \end{aligned}$$

We have taken the master and slave system and these systems have the external disturbances. The external disturbances may be represented as $\omega(t) = \frac{1}{1+5t}$, $t \geq 0$ and the slave system is under the feedback control

$$K = \begin{bmatrix} 0.24 \\ 1.26 \\ 0.26 \end{bmatrix}.$$

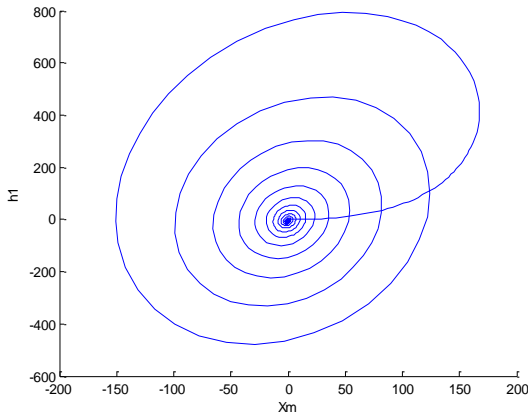
The K is the gain matrix of these systems, the $m=1$ and matrix $A - KCT$ is Hurwitz. The value of A on the imaginary axis has no eigenvalue and matrices (A, B) is controllable. So LMIs (7)–(8) are tested as achievable

$$\begin{aligned} W &= \begin{bmatrix} 8.6722 & -0.3564 \\ -0.3564 & 1.8269 \end{bmatrix}, \\ k &= 1.1837, l = 0.6251, Y = 0.0058. \end{aligned}$$

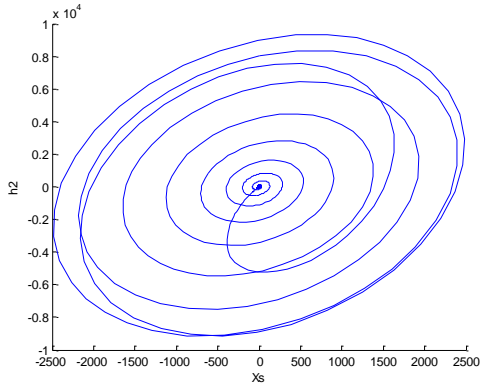
The simulation results of the master and slave system are obtained by different initial values of the system such as

$$\begin{aligned} x(0) &= [-0.4 \quad -0.5]^T, \quad Y_1(0) = -0.3, \quad Y_2(0) = 0.5 \quad \text{and} \\ y(0) &= [-0.30 \quad 0.23]^T. \end{aligned}$$

We achieved synchronization by the selected controller between the system (1) and (2). Different figures such as demonstrates the chaotic behavior of the Lyapunov function systems and the error between the state of master and slave system. Fig. 1 elaborates on the phase portrait of the master and slave system and Fig. 2 illustrates the phase error between the states of master and slave system. So, it is observed that the synchronization errors become zero after a certain time and this synchronizing error time may increase or decrease by changing the gain factor value.



(a)



(b)

FIGURE 1. Phase portrait of Master and Slave systems respectively (a) and (b)

Similarly the error between the output of master and slave system is shown in figure as given below.

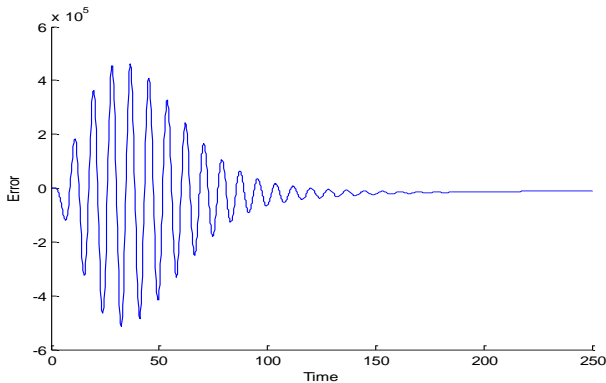


FIGURE 2. Output errors between the Master and Slave system.

VI. CONCLUSION:

Nonlinear systems have disturbance and time delays are present in both states of the master and slave system. In this paper, we have efficiently and precisely achieved the synchronization among the master and slave systems by the robust feedback controller. The close loop error of the system becomes stable by

the performance of the controller and achieves the synchronization between both systems. Due to the efficacy and applicability of the above-proposed methods, it can be widely used in different engineering field applications for numerical solution. In future for research, the dynamics output regulators can be practically used for attaining the H_∞ .

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